Quartic points on the Fermat quartic

$$x^4 + y^4 = 1 \qquad (F_4)$$

Theorem (Ishitsuka, I 10.9091 Tn.-8515 10.66016 1.834 0.84 8G

0 1 -2000.09009728.346 1: cm 14 04 0.84 8G -7.87

Extending Mordell

Suppose $t \in L$. Recall

$$t = \frac{1-x^2}{y^2}, \qquad x^2 = \frac{1-t^2}{1+t^2}, \qquad y^2 = \frac{2t}{1+t^2}$$

Either $x \in L, y \in L$ or $x/y \in L$

If $x \in L$ then $u^2 = (1-t^2)(1+t^2), \qquad u \in L$

This is isomorphic to the elliptic curve E with Cremona label 32a1

Taking the pre-image of points in $E(L) = E(\mathbb{Q}) = \mathbb{Z}/4\mathbb{Z}$ gives us points on F_4 over $\mathbb{Q}$ and $\mathbb{Q}(i)$

Similar computations in the other cases give us points on F_4 over $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$

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Extending Mordell

Suppose $t\ 2\ K; t\ 62\ L$

Let $\text{minpol}_L(t) = t^2 + \quad t + \quad ; \quad\quad ; \quad 2\ L$

Let $A = (1 + t^2)xy$. We can also write

$$A = \quad + \quad t; \quad\quad ; \quad 2\ L$$

We can square both expressions for A:

$$(\quad + \quad t)^2 - 2t(1 - t^2) = \text{minpol}_L(t)(\quad + \quad t) \qquad (1)$$

We get a point $(\quad + \quad (- \quad = \quad); - \quad = \quad)$ on the elliptic curve

$$E : Y^2 = -2X^3 + 2X$$

de ned over L; E is the elliptic curve with Cremona label 64a1!

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Extending Mordell

Squaring and equating the above leads to

$$(\ \ +\ \ t)^2 \quad 2t(1 \quad t^2) = \mathrm{minpol}_L(t)(\ \ +\ \ t) \qquad (2)$$

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Let $B = (1 + \ t^2)y$. We similarly get a L-point on the elliptic curve with Cremona label 32a1

Recall that both elliptic curves have nite rank over L

This gives us nitely many possibilities for hence nitely many equations to solve

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Recall that both elliptic curves have finite rank over L

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Solving these equations gives us points on E_4 defined over $\mathbb{Q}(\sqrt{2}, \sqrt{\;\;7})$ and $\mathbb{Q}(\sqrt{2}, i)$

Other quartic points on the Fermat quartic

Pedro-Jose Cazorla Garcia pointed out that

$$(\sqrt{3})^4 + 2^4 = (\sqrt{5})^4$$

- The elliptic curve 32a1 has rank 0 over $\mathbb{Q}(\sqrt{3})$ and the elliptic curve 64a1 has rank 1 over $\mathbb{Q}(\sqrt{3})$
- The elliptic curve 32a1 has rank 1 over $\mathbb{Q}(\sqrt{5})$ and the elliptic curve 64a1 has rank 0 over $\mathbb{Q}(\sqrt{5})$
- Mordell's strategy won't work here!

Other quartic points on the Fermat quartic